Proofs

Theorem (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the *binomial coefficients*

Lemma

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k\cdot 1} + \frac{1}{1\cdot(n-k+1)}\right) \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n-k+1+k}{k(n-k+1)} \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n+1}{k((n+1)-k)} \\
&= \frac{(n+1)!}{k!((n+1)-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

Proof By Induction

Proof: (By induction)

Base Case: ($n = 0$).

Consider $n = 0$, ie $LHS = (x+y)^0 = 1$ and $RHS = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k = \binom{0}{0} x^0 y^0 = 1$. Therefore it is true for $n = 0$.

Inductive step.

Suppose it is true for $n = k$, ie $(x+y)^k = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r$, then consider $n = k+1$

$$\begin{aligned}
(x+y)^{k+1} &= (x+y)^k (x+y) \\
&= \left(\sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r \right) (x+y) \\
&= \sum_{r=0}^{k} \binom{k}{r} x^{k-r+1} y^r + \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^{r+1} \\
&= \binom{k}{0} x^{k+1} y^0 + \sum_{r=1}^{k} \binom{k}{r} x^{k+1-r} y^r + \sum_{r=1}^{k+1} \binom{k}{r-1} x^{k-(r-1)} y^r \\
&= \binom{k}{0} x^{k+1} y^0 + \sum_{r=1}^{k} \binom{k}{r} x^{k+1-r} y^r + \sum_{r=1}^{k} \binom{k}{r-1} x^{k+1-r)} y^r + \binom{k}{k} x^0 y^{k+1} \\
&= \binom{k}{0} x^{k+1} y^0 + \sum_{r=1}^{k} \left(\binom{k}{r} + \binom{k}{r-1} \right) x^{k+1-r} y^r + \binom{k}{k} x^0 y^{k+1} \\
&= \binom{k+1}{0} x^{k+1} y^0 + \sum_{r=1}^{k} \binom{k+1}{r} x^{k+1-r} y^r + \binom{k+1}{k+1} x^0 y^{k+1} \\
&= \sum_{r=0}^{k+1} \binom{k+1}{r} x^{k+1-r} y^r
\end{aligned}$$

Therefore our statement is true for $n = k+1$.

Since our statement is true for $n = 0$ and if it is true for $n = k$ it is true for $n = k+1$ it is true for $n \geq 0$ by the principle of mathematical induction.

Proof By Combinatorics

Example

Find the first 4 terms of $(2x-3)^6$ in ascending powers of x

$$\begin{aligned}(2x-3)^6 &= {}^6C_0(2x)^0(-3)^6 + {}^6C_1(2x)^1(-3)^5 + {}^6C_2(2x)^2(-3)^4 + {}^6C_3(2x)^3(-3)^3 + o(x^4)\\ &= 1\cdot 1\cdot 729 + 6\cdot 2x\cdot(-243) + \frac{6\cdot 5}{2\cdot 1}\cdot 4x^2\cdot 81 + \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}\cdot 8x^3\cdot(-27) + o(x^4)\\ &= 729 - 2916x + 9720x^2 - 4320x^3 + o(x^4)\end{aligned}$$

Example

In the binomial expansion of

$$(1+kx)^6$$

where k is constant, the coefficient of x^3 is twice as large as the coefficient of x^2. Find the value of k

$$\begin{aligned}\textit{coefficient of } x^3 &= {}^6C_3k^3\\ &= \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}k^3\\ &= 20k^3\\ \textit{coefficient of } x^2 &= {}^6C_2k^2\\ &= \frac{6\cdot 5}{2\cdot 1}k^2\\ &= 15k^2\\ \Rightarrow \quad 20k^3 &= 2\cdot 15k^2\\ \Rightarrow \quad k &= \frac{3}{2}\end{aligned}$$

Example

Calculate the coefficient of x^2 in the expansion of $(2x+1)^3(2-3x)^4$

$$\begin{aligned}
(2x+1)^3 &= {}^3C_0 \cdot 1^3 \cdot (2x)^0 + {}^3C_1 \cdot 1^2 \cdot (2x)^1 + {}^3C_2 \cdot 1^1 \cdot (2x)^2 + o(x^3) \\
&= 1 + 6x + 12x^2 + o(x^3) \\
(2-3x)^4 &= {}^4C_0 \cdot 2^4 \cdot (-3x)^0 + {}^4C_1 \cdot 2^3 \cdot (-3x)^1 + {}^4C_2 \cdot 2^2 \cdot (-3x)^2 + o(x^3) \\
&= 16 - 96x + 216x^2 + o(x^3) \\
\Rightarrow \qquad (2x+1)^3(2-3x)^4 &= (1 + 6x + 12x^2 + o(x^3))(16 - 96x + 216x^2 + o(x^3)) \\
&= \cdots + (1 \cdot 216 - 6 \cdot 96 + 12 \cdot 16)x^2 + \cdots \\
&= \cdots - 168x^2 + \cdots
\end{aligned}$$

Extra hard problem: Show that the coefficient of x is 0 *without doing any calculation*

Example

Find the number of pairs of positive integers x, y which solve the equation:

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 1,000,000$$

Note this expansion looks suspiciously like $(x+2y)^3$, in fact it is, so we want $x + 2y = 100$, or $x = 100 - 2y$, so $y \in \{1, 2, \cdots, 49\}$